SYMMETRY CLASSES OF TENSORS ASSOCIATED WITH PRINCIPAL INDECOMPOSABLE CHARACTERS AND OSIMA IDEMPOTENTS

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Abstract. Two types of symmetry classes of tensors are studied. The first is the symmetry class associated with a principle indecomposable character of the dihedral group. The second is the symmetry class associated with the Osima idempotent corresponding to a p-block of the dihedral group. In each case necessary and sufficient conditions are given for the existence of an orthogonal basis consisting of standard (decomposable) symmetrized tensors.

0. Introduction

The usual notion of symmetry class of tensors involves an ordinary irreducible character χ of a subgroup G of a symmetric group S_m. By definition this symmetry class V_χ is the product $s_\chi V^{\otimes m}$, where s_χ is the central idempotent of the group algebra $\mathbf{C}G$ corresponding to the conjugate character $\overline{\chi}$ and V is a finite-dimensional vector space over the field $\mathbf{C}$ of complex numbers.

One can replace s_χ by any element s of $\mathbf{C}G$ to get a new notion of symmetry class of tensors. Modular representation theory of the group G, which requires the choice of a prime number p, provides three natural choices for such an s: (1) s_φ with φ an irreducible Brauer character of G, (2) s_Φ with Φ a principal indecomposable character of G, and (3) the Osima idempotent s_B corresponding to a p-block B of G. Each of these choices results in a symmetry class of tensors that generalizes the usual notion in the sense that, if p does not divide the order of G, then the symmetry class is the same as that corresponding to an ordinary irreducible character of G (see Remark 2.2).

The first of these three generalizations is the focus of [HK13]. In that paper necessary and sufficient conditions are given under which the symmetry class of tensors associated with an irreducible Brauer character of the dihedral group D_m (viewed as a subgroup of S_m) is guaranteed to have an orthogonal basis consisting of standard (decomposable) symmetrized tensors – a so-called "o-basis."

2010 *Mathematics Subject Classification.* 15A69, 20B05, 20C20, 20C15.

Key words and phrases. Symmetry class of tensors, principal indecomposable character, Osima idempotent.

In this paper we take up the two remaining generalizations, namely, the symmetry class of tensors associated with a principle indecomposable character of G and the symmetry class of tensors associated with the Osima idempotent corresponding to a p-block of G. Taking G to be the dihedral group again we give necessary and sufficient conditions for the existence of an o-basis in each case.

Results about o-bases of symmetry classes of tensors can be found in [BPR03, DP99, DdST05, Hol95, Hol04, HT92, SAJ01]. Symmetry classes of tensors associated with modular characters (i.e., characters having values in a field of prime characteristic) were studied in [SS99].

1. Character theory

Let G be a finite group and fix a prime number p. An element of G is *p-regular* if its order is not divisible by p. Denote by $\hat{G}$ the set of all p-regular elements of G. Let $\mathrm{IBr}(G)$ denote the set of irreducible Brauer characters of G. (A Brauer character is a certain function from $\hat{G}$ to $\mathbf{C}$ associated with an FG-module where F is a suitably chosen field of characteristic p. The Brauer character is irreducible if the associated module is simple. For the theory of Brauer characters, see [Isa94, Ser77, CR62, Fei82].)

Let $\mathrm{Irr}(G)$ denote the set of irreducible characters of G. (Unless preceded by the word "Brauer," the word "character" always refers to an ordinary character.) If the order of G is not divisible by p, then $\hat{G} = G$ and $\mathrm{IBr}(G) = \mathrm{Irr}(G)$.

For a character χ of G denote by $\hat{\chi} : \hat{G} \to \mathbf{C}$ the restriction of χ to $\hat{G}$. For each $\chi \in \mathrm{Irr}(G)$ we have

$$\hat{\chi} = \sum_{\varphi \in \mathrm{IBr}} d_{\varphi\chi}\varphi$$

for some nonnegative integers $d_{\varphi\chi}$, called the *decomposition numbers* of G with respect to the prime number p.

Corresponding to each $\varphi \in \mathrm{IBr}(G)$ is a principal indecomposable character $\Phi_\varphi : G \to \mathbf{C}$. (This is a character of G having the property that its restriction $\hat{\Phi}_\varphi$ to $\hat{G}$ is the Brauer character corresponding to the projective cover of the simple FG-module corresponding to φ.)

1.1 Theorem. [Fei82, IV, 2.5, 3.1] *For every $\varphi \in \mathrm{IBr}(G)$,*

(i) $\Phi_\varphi(g) = 0$ *for all $g \in G\backslash\hat{G}$,*
(ii) $\Phi_\varphi = \sum_{\chi \in \mathrm{Irr}(G)} d_{\phi\chi}\chi$*, where the $d_{\phi\chi}$ are the decomposition numbers of G with respect to the prime number p.*

If p does not divide $|G|$, then the matrix $[d_{\varphi\chi}]$ is the identity matrix and $\Phi_\varphi = \varphi$ for each $\varphi \in \mathrm{IBr}(G)$ $(= \mathrm{Irr}(G))$.

The *Brauer graph* of G (with respect to the prime number p) is the graph with vertex set $\mathrm{Irr}(G)$ and with an edge joining the vertices χ and ψ if and only if there exists $\varphi \in \mathrm{IBr}(G)$ such that $d_{\varphi\chi}$ and $d_{\varphi\psi}$ are both nonzero.

A *p-block* of G is a subset B of $\mathrm{Irr}(G) \cup \mathrm{IBr}(G)$ with the following properties [Isa94, 15.17, 15.27]:

(i) $B \cap \mathrm{Irr}(G)$ is a connected component of the Brauer graph of G,
(ii) $B \cap \mathrm{IBr}(G) = \{\varphi \in \mathrm{IBr}(G) \,|\, d_{\varphi\chi} \neq 0 \text{ for some } \chi \in B \cap \mathrm{Irr}(G)\}$.

The set $\mathrm{Bl}(G)$ of p-blocks of G is a partition of the set $\mathrm{Irr}(G) \cup \mathrm{IBr}(G)$. If p does not divide $|G|$, then each p-block of G is a singleton $\{\chi\}$ with $\chi \in \mathrm{Irr}(G)$.

Let $B \in \mathrm{Bl}(G)$. The *Osima idempotent* corresponding to B is the element s_B of the group algebra $\mathbf{C}G$ given by

$$s_B = \frac{1}{|G|} \sum_{g \in G} \sum_{\chi \in B \cap \mathrm{Irr}(G)} \chi(e)\overline{\chi(g)}g$$

or, equivalently,

$$s_B = \frac{1}{|G|} \sum_{g \in \hat{G}} \sum_{\varphi \in B \cap \mathrm{IBr}(G)} \Phi_\varphi(e)\overline{\varphi(g)}g,$$

where e is the identity element of G and bar denotes conjugation [Isa94, 15.21, 15.26, 15.30]. It follows from the second expression that the sum over $g \in G$ in the first expression can be replaced by the sum over $g \in \hat{G}$. The elements s_B with $B \in \mathrm{Bl}(G)$ are pairwise orthogonal central idempotents of the algebra $\mathbf{C}G$ and they sum to 1. That is to say, for $B, B' \in \mathrm{Bl}(G)$ we have $s_B s_{B'} = \delta_{BB'}$ (Kronecker delta), $\sum_{B \in \mathrm{Bl}(G)} s_B = 1$, and each s_B commutes with every element of $\mathbf{C}G$.

2. Symmetry classes of tensors

Fix positive integers m and n and put $\Gamma_{m,n} = \{\gamma \in \mathbf{Z}^m \,|\, 1 \leq \gamma_i \leq n\}$. Fix a subgroup G of the symmetric group S_m of degree m. A right action of G on the set $\Gamma_{m,n}$ is given by $\gamma\sigma = (\gamma_{\sigma(1)}, \ldots, \gamma_{\sigma(m)})$ ($\gamma \in \Gamma_{m,n}, \sigma \in G$). For $\gamma \in \Gamma_{m,n}$ put $G_\gamma = \{\sigma \in G \,|\, \gamma\sigma = \gamma\}$, the stabilizer of γ.

Let V be a complex inner product space of dimension n and let $\{e_i \,|\, 1 \leq i \leq n\}$ be an orthonormal basis for V. The inner product on V induces an inner product on $V^{\otimes m}$ (the mth tensor power of V) and, with respect to this inner product, the set $\{e_\gamma \,|\, \gamma \in \Gamma_{m,n}\}$ is an orthonormal basis for $V^{\otimes m}$, where $e_\gamma = e_{\gamma_1} \otimes \cdots \otimes e_{\gamma_m}$.

The space $V^{\otimes m}$ is a left $\mathbf{C}G$-module with action given by $\sigma e_\gamma = e_{\gamma\sigma^{-1}}$ ($\sigma \in G, \gamma \in \Gamma_{m,n}$), extended linearly. The inner product on $V^{\otimes m}$ is G-invariant, which is to say $(\sigma v, \sigma w) = (v, w)$ for all $\sigma \in G$ and all $v, w \in V^{\otimes m}$.

Let $s \in \mathbf{C}G$. We have $s = \sum_{\sigma \in G} a_\sigma \sigma$ for some $a_\sigma \in \mathbf{C}$. For $\gamma \in \Gamma_{m,n}$ the *standard* (or *decomposable*) *symmetrized tensor* corresponding to s and γ is given by

$$e_\gamma^s = s e_\gamma = \sum_{\sigma \in G} a_\sigma e_{\gamma\sigma^{-1}}.$$

The *symmetry class of tensors* associated with s is

$$V_s = sV^{\otimes m} = \langle e^s_\gamma \,|\, \gamma \in \Gamma_{m,n}\rangle.$$

The *orbital subspace* of V_s corresponding to $\gamma \in \Gamma_{m,n}$ is

$$V^s_\gamma = sV_\gamma = \langle e^s_{\gamma\sigma} \,|\, \sigma \in G\rangle,$$

where $V_\gamma = \langle e_{\gamma\sigma} \,|\, \sigma \in G\rangle$.

An *o-basis* of a subspace W of V_s is an orthogonal basis of W of the form $\{e^s_{\gamma_1}, e^s_{\gamma_2}, \dots, e^s_{\gamma_t}\}$ for some $\gamma_i \in \Gamma_{m,n}$. By convention, the empty set is regarded as an o-basis of the zero subspace of V_s.

Let $\Delta = \Delta_G$ be a set of representatives of the orbits of $\Gamma_{m,n}$ under the action of G.

2.1 Theorem. [HK13, 1.1] *For every $s \in \mathbf{C}G$,*

$$V_s = \dot{\sum_{\gamma\in\Delta}} V^s_\gamma \quad (\textit{orthogonal direct sum}).$$

In particular, V_s has an o-basis if and only if V^s_γ has an o-basis for each $\gamma \in \Delta$.

Let S be a subset of G containing the identity permutation 1 and let $\varphi : S \to \mathbf{C}$ be a function. (For instance, φ could be a character of G, in which case $S = G$, or φ could be a Brauer character of G, in which case $S = \hat{G}$.) Put

$$s_\varphi = \frac{\varphi(1)}{|S|} \sum_{\sigma\in S} \varphi(\sigma)\sigma \in \mathbf{C}G.$$

Let $\gamma \in \Gamma_{m,n}$. We write $e^{s_\varphi}_\gamma$, V_{s_φ}, and $V^{s_\varphi}_\gamma$ more simply as e^φ_γ, V_φ, and V^φ_γ, respectively. Similarly, for $B \in \mathrm{Bl}(G)$ we write $e^{s_B}_\gamma$, V_{s_B}, and $V^{s_B}_\gamma$ more simply as e^B_γ, V_B, and V^B_γ, respectively.

2.2 *Remark.* Let $\varphi \in \mathrm{IBr}(G)$ and let B be the p-block of G containing φ. Each of the symmetry classes of tensors V_φ, V_{Φ_φ}, and V_B can be regarded as a generalization of the classical symmetry class V_χ associated with an ordinary irreducible character χ of G. Indeed, if p does not divide $|G|$, then φ is an ordinary irreducible character and we have $\Phi_\varphi = \varphi$ and $B = \{\varphi\}$, so these three symmetry classes reduce to the classical one. (In this case $s_B = s_{\overline{\varphi}}$ so $V_B = V_{\overline{\varphi}}$. The conjugation here could be avoided by replacing $\varphi(\sigma)$ by $\overline{\varphi(\sigma)}$ in the definition of the symmetrizer s_φ. This would be more natural since then s_φ would be the central idempotent of $\mathbf{C}G$ corresponding to the character φ, but in the literature the symmetrizer s_φ is usually defined as above.)

2.3 Theorem. [Mer97, 6.6] *We have*

$$V^{\otimes m} = \dot{\sum_{\chi \in \mathrm{Irr}(G)}} V_\chi \qquad (\textit{orthogonal direct sum}).$$

This theorem fails to hold if $\mathrm{Irr}(G)$ is replaced by $\mathrm{IBr}(G)$ [HK13, Example 2.5].

Fix a character ψ of G. We have $\psi = \sum_{i=1}^{t} m_i \chi_i$ with the m_i positive integers and with $\chi_1, \ldots, \chi_t$ distinct irreducible characters of G, called the irreducible constituents of ψ.

2.4 Theorem. *Let ψ be as above. For every $\gamma \in \Gamma_{m,n}$ and $\sigma \in G$,*

$$(e^\psi_{\gamma\sigma}, e^\psi_\gamma) = \frac{\psi(1)^2}{|G|} \sum_{i=1}^{t} \frac{m_i^2}{\chi_i(1)} \sum_{\tau \in G_\gamma} \chi_i(\tau\sigma).$$

Proof. First,

$$\begin{aligned} s_\psi &= \frac{\psi(1)}{|G|} \sum_{\tau \in G} \psi(\tau)\tau = \frac{\psi(1)}{|G|} \sum_{\tau \in G} \sum_{i=1}^{t} m_i \chi_i(\tau)\tau \\ &= \psi(1) \sum_{i=1}^{t} \frac{m_i}{\chi_i(1)} \frac{\chi_i(1)}{|G|} \sum_{\tau \in G} \chi_i(\tau)\tau = \psi(1) \sum_{i=1}^{t} \frac{m_i}{\chi_i(1)} s_{\chi_i}. \end{aligned}$$

Therefore,

$$\begin{aligned} (e^\psi_{\gamma\sigma}, e^\psi_\gamma) &= (s_\psi e_{\gamma\sigma}, s_\psi e_\gamma) = \psi(1)^2 \sum_{i=1}^{t} \sum_{j=1}^{t} \frac{m_i m_j}{\chi_i(1)\chi_j(1)} (e^{\chi_i}_{\gamma\sigma}, e^{\chi_j}_\gamma) \\ &= \psi(1)^2 \sum_{i=1}^{t} \frac{m_i^2}{\chi_i(1)^2} (e^{\chi_i}_{\gamma\sigma}, e^{\chi_i}_\gamma) = \frac{\psi(1)^2}{|G|} \sum_{i=1}^{t} \frac{m_i^2}{\chi_i(1)} \sum_{\tau \in G_\gamma} \chi_i(\tau\sigma), \end{aligned}$$

where the third equality uses that V_{χ_i} is orthogonal to V_{χ_j} for $i \neq j$ (Theorem 2.3) and the last equality is from [Fre73, p. 339]. □

2.5 Theorem. [HK13, 1.6] *Let ψ be as above. For every $\gamma \in \Gamma_{m,n}$,*

$$\dim V^\psi_\gamma = \sum_{i=1}^{t} \dim V^{\chi_i}_\gamma.$$

The summands in the preceding theorem are given by the following well-known formula due to Freese.

2.6 Theorem. [Fre73, p. 339] *For every $\chi \in \mathrm{Irr}(G)$ and $\gamma \in \Gamma_{m,n}$,*

$$\dim V^\chi_\gamma = \frac{\chi(1)}{|G_\gamma|} \sum_{\sigma \in G_\gamma} \chi(\sigma) = \chi(1)(\chi, 1)_{G_\gamma}.$$

3. The dihedral group

Assume that $m \geq 3$ and define $r, s \in S_m$ by

$$r = \begin{pmatrix} 1 & 2 & 3 & \cdots & m-1 & m \\ 2 & 3 & 4 & \cdots & m & 1 \end{pmatrix}, \quad s = \begin{pmatrix} 1 & 2 & 3 & \cdots & m-1 & m \\ 1 & m & m-1 & \cdots & 3 & 2 \end{pmatrix}.$$

Then $G = D_m = \langle r, s \rangle$ is the *dihedral group* of degree m. The elements r and s satisfy the relations $r^m = 1$, $s^2 = 1$, and $sr^k s = r^{-k}$ for all k. It follows that $G = \{r^k, sr^k \,|\, 0 \leq k < m\}$. The order of G is $2m$ and the cyclic subgroup C of G generated by r has order m.

The ordinary irreducible characters of G are [Ser77, pp. 37–38]

	r^k	sr^k	
ψ_1	1	1	
ψ_2	1	-1	
ψ_3	$(-1)^k$	$(-1)^k$	(m even)
ψ_4	$(-1)^k$	$(-1)^{k+1}$	(m even)
χ_j	$2\cos\dfrac{2\pi kj}{m}$	0	$(1 \leq j < m/2)$

Table 1.

Let p be a fixed prime number. We have $m = p^q \ell$ for a nonnegative integer q and a positive integer ℓ not divisible by p. Define

$$\varepsilon = \begin{cases} 4, & l \text{ even}, p \neq 2, \\ 2, & l \text{ odd}, p \neq 2, \\ 1, & p = 2. \end{cases}$$

For each $1 \leq i \leq \varepsilon$ and $1 \leq j < \ell/2$ put

$$\varphi_i^1 = \hat{\psi}_i, \qquad \varphi_j^2 = \hat{\chi}_j.$$

3.1 Theorem [HK13, 2.1]. *The complete list of irreducible Brauer characters of G is φ_i^1 $(1 \leq i \leq \varepsilon)$, φ_j^2 $(1 \leq j < \ell/2)$.* □

(The superscript in the notation corresponds to the degree of the Brauer character.)

Put $J = \{j \in \mathbf{Z} \,|\, 1 \leq j < m/2\}$ and note that $\{\chi_j \,|\, j \in J\}$ is the set of degree two irreducible characters of G. For $i \in \mathbf{Z}$ put

$$J_i = J \cap \{k\ell \pm i \,|\, k \in \mathbf{Z}\}.$$

3.2 Theorem. *The sets J_i, $0 \leq i \leq \ell/2$, form a partition of the set J.*

Proof. Put $U = \bigcup_{i=0}^{\ell/2} J_i$. From the definition of J_i we have $U \subseteq J$. Let $j \in J$. By the division algorithm there exist integers k and t with $0 \leq t < \ell$ such that $j = k\ell + t$. If $t \leq \ell/2$, then $j \in J_t \subseteq U$. If $t > \ell/2$, then putting $i = \ell - t$ we have $0 < i < \ell/2$ and $j = k\ell + (\ell - i) = (k+1)\ell - i \in J_i \subseteq U$. Therefore, $J = U$.

Next we check that the sets J_i are pairwise disjoint. Let $0 \leq i \leq i' \leq \ell/2$ and let $j \in J_i \cap J_{i'}$. Then $k\ell \pm i = j = k'\ell \pm i'$ for some integers k and k'. If both signs are plus, we get $(k-k')\ell = i' - i$, so $0 \leq (k-k')\ell \leq \ell/2$ implying $0 \leq k - k' \leq 1/2$, whence $k' = k$ and $i' = i$. If both signs are minus, a similar argument again yields $i' = i$. Now suppose $k\ell + i = k'\ell - i'$. Then $i + i' = (k'-k)\ell$, so $0 \leq (k'-k)\ell \leq \ell$, implying $i + i'$ is 0 or ℓ. The first case yields $i' = 0 = i$; the latter yields $i' = \ell/2 = i$. The remaining case $k\ell - i = k'\ell + i'$ is handled similarly. □

The decomposition numbers $d_{\varphi\psi}$ of G can be read off from the equations in the following theorem.

3.3 Theorem.

(i) *For* $1 \leq i \leq \eta$,

$$\hat{\psi}_i = \begin{cases} \varphi_1^1, & \text{if } p = 2, \\ \varphi_i^1, & \text{if } p \neq 2, \end{cases}$$

where η *is* 2 *or* 4 *according as* m *is odd or even.*

(ii) *For* $1 \leq j < m/2$ *we have* $j \in J_i$ *for a unique* $0 \leq i \leq \ell/2$ *and*

$$\hat{\chi}_j = \begin{cases} \begin{cases} 2\varphi_1^1, & \text{if } p = 2, \\ \varphi_1^1 + \varphi_2^1, & \text{if } p \neq 2, \end{cases} & \text{if } i = 0, \\ \varphi_i^2, & \text{if } 0 < i < \ell/2, \\ \varphi_3^1 + \varphi_4^1, & \text{if } i = \ell/2. \end{cases}$$

Moreover, for each $0 \leq i \leq \ell/2$ *and* $j \in J_i$ *the character* $\hat{\chi}_j$ *is as indicated.*

Proof. (i) If $p \neq 2$ the claim follows from the definition of φ_i^1. Assume $p = 2$. For each $1 \leq i \leq \eta$ the character ψ_i is linear (i.e., of degree one) so that $\hat{\psi}_i$ is linear as well. Since φ_1^1 is the only linear Brauer character of G in this case the claim follows.

(ii) The sets J_i, $0 \leq i \leq \ell/2$, form a partition of the set $J = \{j \in \mathbf{Z} \mid 1 \leq j < m/2\}$ (Theorem 3.2), so it suffices to prove the last sentence. Let $0 \leq i \leq \ell/2$ and let $j \in J_i$. It follows from Table 1 that in all cases both sides of the equation vanish on $\hat{G} \backslash \hat{C} = \hat{G} \backslash C$ (this set being empty when $p = 2$), so it suffices to show that both sides agree on $\hat{C} = \langle r^{p^q} \rangle$. We have $j = k\ell \pm i$ for some integer k. Using the cosine sum and difference formulas we get for each integer t

$$\begin{aligned}\hat{\chi}_j(r^{p^q t}) &= 2\cos\left(\frac{2\pi p^q t(k\ell \pm i)}{m}\right)\\ &= 2\cos(2\pi tk)\cos\left(\frac{2\pi p^q ti}{m}\right) \mp \sin(2\pi tk)\sin\left(\frac{2\pi p^q ti}{m}\right)\\ &= 2\cos\left(\frac{2\pi p^q ti}{m}\right)\\ &= \begin{cases} 2, & \text{if } i = 0,\\ \varphi_i^2(r^{p^q t}), & \text{if } 0 < i < \ell/2,\\ 2(-1)^{p^q t}, & \text{if } i = \ell/2.\end{cases}\end{aligned}$$

(For the case $i = \ell/2$ we have used the fact that ℓ is even so that p is not 2 and is therefore odd.) Therefore, both sides of the equation agree on $\hat{C}$ and the proof is complete. □

We write Φ_i^j for the principal indecomposable character Φ_φ corresponding to the irreducible Brauer character $\varphi = \varphi_i^j$.

3.4 Theorem.

(i) *If $p = 2$, then*

$$\Phi_1^1 = \sum_{j=1}^{\eta} \psi_j + 2\sum_{j\in J_0} \chi_j,$$

and if $p \neq 2$, then

$$\Phi_i^1 = \begin{cases} \psi_i + \sum_{j\in J_0} \chi_j, & i = 1, 2,\\ \psi_i + \sum_{j\in J_{\ell/2}} \chi_j, & i = 3, 4 \quad (m \textit{ even}),\end{cases}$$

where η is 2 or 4 according as m is odd or even.

(ii) *For each $1 \le i < \ell/2$,*

$$\Phi_i^2 = \sum_{j\in J_i} \chi_j.$$

Proof. According to Theorem 1.1, for $\varphi \in \mathrm{IBr}(G)$ the multiplicity of an irreducible character χ of G as a summand of Φ_φ is the decomposition number $d_{\varphi\chi}$, which is the same as the multiplicity of φ as a summand of $\hat{\chi}$.

Assume that $p = 2$. By Theorem 3.3, we see that φ_1^1 appears with multiplicity 1 in each of the characters $\hat{\psi}_i$, $1 \le i \le \eta$, it appears with multiplicity 2 in each of the characters $\hat{\chi}_j$, $j \in J_0$, and it does not appear in any of the other restricted irreducible characters. Therefore, by the preceding paragraph the decomposition of Φ_1^1 is as stated in (i).

The rest of the formulas are verified in a similar fashion. □

3.5 Theorem.

(i) *For* $1 \leq i \leq \varepsilon$ *the space* $V_{\Phi_i^1}$ *has an o-basis if and only if at least one of the following holds:*
   (a) $\dim V = 1$,
   (b) $p = 2$,
   (c) m *is not divisible by* p.

(ii) *For* $1 \leq i < \ell/2$ *the space* $V_{\Phi_i^2}$ *has an o-basis if and only if either* $\dim V = 1$ *or* $\ell' \equiv 0 \mod 4$, *where* $\ell' = \ell/\gcd(\ell, i)$.

Proof. (i) Fix $1 \leq i \leq \varepsilon$.

If $\dim V = 1$, then $V_{\Phi_i^1} = \langle e_\gamma^{\Phi_i^1} \rangle$, where $\gamma = (1, 1, \ldots, 1)$, so $V_{\Phi_i^1}$ has o-basis $\{e_\gamma^{\Phi_i^1}\}$ or $\emptyset$ according as $\dim V_{\Phi_i^1}$ is 1 or 0.

Assume $p = 2$ so that $\varepsilon = 1$. In this case, $\hat{G}$ equals $\langle r^{p^q} \rangle$, a subgroup of G. Let λ be the trivial character of $\hat{G}$. By Theorem 3.3 we have $\hat{\psi}_i = \varphi_1^1$ for all $1 \leq i \leq \eta$, and $\hat{\chi}_j = 2\varphi_1^1$ for all $j \in J_0$. Since $\varphi_1^1 = \hat{\psi}_1 = \lambda$ it follows from Theorem 3.4 that $s_{\Phi_1^1}$ is a nonzero scalar multiple of s_λ. In particular, $V_{\Phi_1^1} = V_\lambda$. Now each orbital subspace V_γ^λ has dimension at most one (Theorem 2.6) and therefore has an o-basis. By Theorem 2.1, V_λ has an o-basis and therefore $V_{\Phi_1^1}$ does as well.

Assume that m is not divisible by p. If $p = 2$, then $V_{\Phi_i^1}$ has an o-basis as we have just seen, so assume that $p \neq 2$. Then $\hat{G} = G$ and hence $\Phi_i^1 = \psi_i$. In particular, $V_{\Phi_i^1} = V_{\psi_i}$. Since ψ_i is of degree one, each orbital subspace $V_\gamma^{\psi_i}$ has dimension at most one and hence $V_{\Phi_i^1}$ has an o-basis by an argument similar to that in the preceding paragraph.

Now assume that none of the three conditions stated in the theorem holds. Let $\gamma = (1, 2, \ldots, 2)$, which is in $\Gamma_{m,n}$ since $n = \dim V \geq 2$. Note that $G_\gamma = \{1, s\}$.

Claim: $(e_{\gamma\sigma}^{\Phi_i^1}, e_\gamma^{\Phi_i^1}) \neq 0$ for every $\sigma \in G$. Let $\sigma \in G$. Since $G = C \cup sC$ and $s \in G_\gamma$ we may assume that $\sigma \in C$. Put

$$T_i = \begin{cases} J_0, & \text{if } i \in \{1, 2\}, \\ J_{\ell/2}, & \text{if } i \in \{3, 4\}. \end{cases}$$

(If $i \in \{3, 4\}$, then m is even and, since $p \neq 2$, it follows that ℓ is even as well so that $J_{\ell/2}$ is defined.) By Theorems 2.4 and 3.4 we get

$$c(e_{\gamma\sigma}^{\Phi_i^1}, e_\gamma^{\Phi_i^1}) = \sum_{\tau \in G_\gamma} \Big(2\psi_i(\tau\sigma) + \sum_{j \in T_i} \chi_j(\tau\sigma) \Big), \tag{3.5.1}$$

where $c = 2|G|/\Phi_i^1(1)^2$. And then using Theorem 3.4 again we get

$$c(e_{\gamma\sigma}^{\Phi_i^1}, e_\gamma^{\Phi_i^1}) = \sum_{\tau \in G_\gamma} \Big(\psi_i(\tau\sigma) + \Phi_i^1(\tau\sigma) \Big). \tag{3.5.2}$$

Assume that $\sigma \notin \hat{G}$. From Equation (3.5.2) and the fact that principal indecomposable characters vanish on $G \setminus \hat{G}$ (Theorem 1.1), we get

$$\begin{aligned} c(e^{\Phi_i^1}_{\gamma\sigma}, e^{\Phi_i^1}_{\gamma}) &= (\psi_i + \Phi_i^1)(\sigma) + (\psi_i + \Phi_i^1)(s\sigma) \\ &= \psi_i(\sigma) + \big(2\psi_i + \sum_{j \in T_i} \chi_j\big)(s\sigma) = \psi_i(\sigma) + 2\psi_i(s\sigma) \neq 0, \end{aligned}$$

where we have used that $\sigma \in C$ and Table 1.

Now assume that $\sigma \in \hat{G}$ (with the assumption $\sigma \in C$ still in force). For $j \in T_i$, Theorem 3.3 gives

$$\begin{aligned} \chi_j(\sigma) &= \begin{cases} \psi_1(\sigma) + \psi_2(\sigma), & i \in \{1, 2\}, \\ \psi_3(\sigma) + \psi_4(\sigma), & i \in \{3, 4\}, \end{cases} \\ &= 2\psi_i(\sigma), \end{aligned}$$

the second equality from Table 1 and again the fact that $\sigma \in C$. Also, from Table 1 we have $\psi_i(s\sigma) = (-1)^{i+1}\psi_i(\sigma)$. So Equation (3.5.1) gives

$$\begin{aligned} c(e^{\Phi_i^1}_{\gamma\sigma}, e^{\Phi_i^1}_{\gamma}) &= \big(2\psi_i + \sum_{j \in T_i} \chi_j\big)(\sigma) + \big(2\psi_i + \sum_{j \in T_i} \chi_j\big)(s\sigma) \\ &= 2\psi_i(\sigma) + \sum_{j \in T_i} 2\psi_i(\sigma) + 2(-1)^{i+1}\psi_i(\sigma) \\ &= 2\Big(1 + |T_i| + (-1)^{i+1}\Big)\psi_i(\sigma). \end{aligned}$$

Therefore, to show that the inner product is nonzero it is enough to show that T_i is nonempty. First, $p \neq 2$, so $\ell = m/p^q < m/2$, implying that $\ell \in J_0 = T_i$ if $i \in \{1, 2\}$. Now assume that $i \in \{3, 4\}$. Then m is even and, since p is odd, it follows that $\ell = m/p^q$ is even. In particular, $\ell/2$ is an integer. Since $\ell/2 = m/2p^q < m/2$, we have $\ell/2 \in J_{\ell/2} = T_i$. This establishes the claim.

The preceding paragraph shows that the set T_i is nonempty, so there exists $j \in T_i$. By Theorem 2.6 we have $\dim V_\gamma^{\chi_j} = 2$. Now the irreducible character χ_j is a constituent of the character Φ_i^1 by Theorem 3.4, so Theorem 2.5 gives $\dim V_\gamma^{\Phi_i^1} > 1$. Therefore, by our claim above and because the inner product is G-invariant, $V_\gamma^{\Phi_i^1}$ does not have an o-basis. Using Theorem 2.1 we conclude that $V_{\Phi_i^1}$ does not have an o-basis and the proof of (i) is complete.

(ii) Let $1 \leq i < \ell/2$. From Theorem 3.4 and then Theorem 3.3 we get

$$\hat{\Phi}_i^2 = \sum_{j \in J_i} \hat{\chi}_j = \sum_{j \in J_i} \varphi_i^2 = |J_i|\varphi_i^2.$$

Since Φ_i^2 vanishes on $G \backslash \hat{G}$ (Theorem 1.1), it follows that $s_{\Phi_i^2}$ is a nonzero scalar multiple of $s_{\varphi_i^2}$. In particular, $V_{\Phi_i^2} = V_{\varphi_i^2}$ and $V_{\Phi_i^2}$ has an o-basis if and only if $V_{\varphi_i^2}$ has an o-basis. Since $V_{\varphi_i^2}$ has an o-basis if and only if either

$\dim V = 1$ or $\ell' \equiv 0 \mod 4$, where $\ell' = \ell/\gcd(\ell, i)$ [HK13, Theorem 2.3], the proof is complete. □

We now turn to a discussion of the symmetry classes associated with Osima idempotents. This requires first a determination of the p-blocks of G.

Put

$$B_1^1 = \begin{cases} \{\psi_i, \chi_j, \varphi_1^1 \,|\, 1 \leq i \leq \eta, j \in J_0\}, & \text{if } p = 2, \\ \{\psi_i, \chi_j, \varphi_i^1 \,|\, 1 \leq i \leq 2, j \in J_0\}, & \text{if } p \neq 2, \end{cases}$$
$$B_3^1 = \{\psi_i, \chi_j, \varphi_i^1 \,|\, 3 \leq i \leq 4, j \in J_{\ell/2}\} \quad (m \text{ even}),$$
$$B_i^2 = \{\chi_j, \varphi_i^2 \,|\, j \in J_i\}, 1 \leq i < \ell/2,$$

where η is 2 or 4 according as m is odd or even.

3.6 Theorem. *If p does not divide $|G|$, then the p-blocks of G are the singleton sets $\{\chi\}$, $\chi \in \operatorname{Irr}(G)$. If p divides $|G|$, then the p-blocks of G are B_1^1, B_3^1, B_i^2 $(1 \leq i < \ell/2)$.*

Note: The notation B_i^j is used to indicate that φ_i^j is in the p-block and i is the least index for which this is the case.

Proof. It has already been noted (Section 1) that if p does not divide $|G|$, then each p-block of G is a singleton set as claimed.

Assume that p divides $|G|$. The following arguments use Theorem 3.3 repeatedly with no further mention.

Let B be the p-block of G containing φ_1^1.

Assume that $p = 2$. Then $\hat{\psi}_i = \varphi_1^1$ for each $1 \leq i \leq \eta$ and $\hat{\chi}_j = 2\varphi_1^1$ for all $j \in J_0$, so that B contains B_1^1. No ordinary irreducible character not in B_1^1 has the property that its restriction to $\hat{G}$ has φ_1^1 as a constituent, so $B = B_1^1$.

Now assume that $p \neq 2$. Since $p \mid |G| = 2m$, we have $p \mid m$, so $1 \leq \ell < m/2$. Therefore, $\chi_\ell \in \operatorname{Irr}(G)$ and, since $\ell \in J_0$, we have $\hat{\chi}_\ell = \varphi_1^1 + \varphi_2^1$. This implies that B contains χ_ℓ and hence φ_2^1 as well. Now, $\hat{\psi}_i = \varphi_i^1$ $(i = 1, 2)$ and $\hat{\chi}_j = \varphi_1^1 + \varphi_2^1$ for all $j \in J_0$, so B contains B_1^1. No ordinary irreducible character not in B_1^1 has the property that its restriction to $\hat{G}$ has either φ_1^1 or φ_2^1 as a constituent, so $B = B_1^1$.

Assume that m is even and let B be the p-block of G containing φ_3^1. Then $p \neq 2$ and ℓ is even. Since $p \mid |G| = 2m$, we have $p \mid m$, so $\ell < m$ and $1 \leq \ell/2 < m/2$. Therefore, $\chi_{\ell/2} \in \operatorname{Irr}(G)$ and, since $\ell/2 \in J_{\ell/2}$, we have $\hat{\chi}_{\ell/2} = \varphi_3^1 + \varphi_4^1$. This implies that B contains $\chi_{\ell/2}$ and hence φ_4^1 as well. Arguing as in the preceding case, we get $B = B_3^1$.

Finally, let $1 \leq i < \ell/2$ and let B be the p-block of G containing φ_i^2. For each $j \in J_i$ we have $\hat{\chi}_j = \varphi_i^2$, so B contains B_i^2. No ordinary irreducible character not in B_i^2 has the property that its restriction to $\hat{G}$ has φ_i^2 as a constituent, so $B = B_i^2$. This completes the proof. □

3.7 Theorem. *Let B be a p-block of G. If B contains a linear character, then V_B has an o-basis. Otherwise, $B = B_i^2$ for some $1 \leq i < \ell/2$ and V_B has an o-basis if and only if either $\dim V = 1$ or $\ell' \equiv 0 \mod 4$, where $\ell' = \ell/\gcd(\ell, i)$.*

Proof. Assume that B contains a linear character. Assume first that p does not divide $|G|$. Then $B \cap \mathrm{Irr}(G) = \{\psi\}$ for some linear $\psi \in \mathrm{Irr}(G)$, implying s_B is a nonzero scalar multiple of s_ψ. In particular, $V_B = V_\psi$. Each orbital subspace V_γ^ψ has dimension either 0 or 1 and therefore has an o-basis, so that V_B has an o-basis by Theorem 2.1.

Now assume that p divides $|G|$. Then B is either B_1^1 or B_3^1 by Theorem 3.6. Assume that $B = B_1^1$. Then

$$s_B = \frac{1}{|G|} \sum_{\sigma \in \hat{G}} \sum_{\chi \in B \cap \mathrm{Irr}(G)} \chi(1)\overline{\chi(\sigma)}\sigma = \frac{1}{|G|} \sum_{\sigma \in \hat{G}} \left[\sum_{i=1}^{\theta} \psi_i(\sigma) + \sum_{j \in J_0} 2\chi_j(\sigma) \right] \sigma,$$

where

$$\theta = \begin{cases} 4, & \text{if } p = 2 \text{ and } m \text{ is even}, \\ 2, & \text{otherwise} \end{cases}$$

(see comment after the definition of s_B in Section 1 about replacing G by $\hat{G}$). It follows from Table 1 that $\sum_{i=1}^{\theta} \psi_i(\sigma) = 0$ and $\chi_j(\sigma) = 0$ for all $\sigma \in G \backslash C$ and all j, so in the sum above we can replace $\sigma \in \hat{G}$ by $\sigma \in \hat{C}$.

Claim: For each $1 \leq i \leq \theta$ we have $\psi_i|_{\hat{C}} = \lambda$, where λ is the trivial character of $\hat{C}$. When $i \in \{1, 2\}$ this follows immediately from Table 1. Suppose $\theta = 4$ and let $i \in \{3, 4\}$. Then $p = 2$ and m is even. Since $m = p^q \ell = 2^q \ell$ with ℓ odd, it follows that $q > 0$. Therefore $\psi_i(r^{p^q}) = (-1)^{2^q} = 1$ and since $\hat{C} = \langle r^{p^q} \rangle$ and ψ_i is a homomorphism, the claim follows.

For $j \in J_0$, Theorem 3.3 gives

$$\chi_j|_{\hat{C}} = (2\varphi_1^1)|_{\hat{C}} = (2\psi_1)|_{\hat{C}} = 2\lambda$$

if $p = 2$, and

$$\chi_j|_{\hat{C}} = (\varphi_1^1 + \varphi_2^1)|_{\hat{C}} = (\psi_1 + \psi_2)|_{\hat{C}} = 2\lambda$$

if $p \neq 2$. This, together with the above claim, shows that s_B is a nonzero scalar multiple of s_λ. In particular, V_B equals V_λ. Since λ is linear, V_λ has an o-basis (by an argument similar to that in the first paragraph), so V_B does as well.

Now assume that $B = B_3^1$. Then

$$s_B = \frac{1}{|G|} \sum_{\sigma \in \hat{G}} \left[\sum_{i=3}^{4} \psi_i(\sigma) + \sum_{j \in J_{\ell/2}} 2\chi_j(\sigma) \right] \sigma.$$

From Table 1 we see that $\sum_{i=3}^{4} \psi_i(\sigma) = 0$ and $\chi_j(\sigma) = 0$ for all $\sigma \in G \backslash C$ and all j, so in the sum we may replace $\sigma \in \hat{G}$ by $\sigma \in \hat{C}$ as before. Now for $i \in \{3,4\}$ we have $\psi_i|_{\hat{C}} = \mu$, where μ is the character of $\hat{C}$ given by $\mu(r^k) = (-1)^k$. And for $a \in J_{\ell/2}$, Theorem 3.3 gives

$$\chi_j|_{\hat{C}} = (\varphi_3^1 + \varphi_4^1)|_{\hat{C}} = (\psi_3 + \psi_4)|_{\hat{C}} = 2\mu.$$

Therefore, s_B is a nonzero scalar multiple of s_μ. Since μ is linear, we conclude as in the preceding case that V_B has an o-basis.

Finally, assume that B does not contain a linear character. Then Theorem 3.6 gives $B = B_i^2$ for some $1 \le i < \ell/2$ (and this holds even if p does not divide $|G|$, since then $B = \{\chi_i\} = B_i^2$ for some $1 \le i < m/2 = \ell/2$). Now $B \cap \mathrm{Irr}(G) = \{\chi_j \,|\, j \in J_i\}$, and for all $j \in J_i$ we have $\overline{\chi_j} = \chi_j$ (Table 1) and $\hat{\chi}_j = \varphi_i^2$ (Theorem 3.3), so

$$s_B = \frac{1}{|G|} \sum_{\sigma \in \hat{G}} \sum_{j \in J_i} 2\overline{\chi_j(\sigma)}\sigma = \frac{1}{|G|} \sum_{\sigma \in \hat{G}} 2\varphi_i^2(\sigma)\sigma,$$

which is a nonzero scalar multiple of $s_{\varphi_i^2}$. In particular, V_B equals $V_{\varphi_i^2}$ and the claim follows from [HK13, Theorem 2.3]. □

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